ON THE MONODROMY OF THE HITCHIN CONNECTION

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Abstract. For any genus $g \geq 2$ we give an example of a family of smooth complex projective curves of genus g such that the image of the monodromy representation of the Hitchin connection on the sheaf of generalized SL(2)-theta functions of level $l \neq 1, 2, 4$ and 8 contains an element of infinite order.

1. Introduction

Let $\pi : \mathcal{C} \to \mathcal{B}$ be a family of smooth connected complex projective curves of genus $g \geq 2$ parameterized by a smooth complex manifold $\mathcal{B}$. For any integers $l \geq 1$, called the level, and $r \geq 2$ we denote $\mathcal{Z}_l$ the complex vector bundle over $\mathcal{B}$ having fibers $H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(r)), \mathcal{L}^{\otimes l})$, where $\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(r))$ is the moduli space of semistable rank-r vector bundles with trivial determinant over the curve $\mathcal{C}_b = \pi^{-1}(b)$ for $b \in \mathcal{B}$ and $\mathcal{L}$ is the ample generator of its Picard group. Following Hitchin [H], the bundle $\mathcal{Z}_l$ is equipped with a projectively flat connection called the Hitchin connection.

The main result of this paper is the following

Theorem. *Assume that the level $l \neq 1, 2, 4$ and 8 and that the rank $r = 2$. For any genus $g \geq 2$ there exists a family $\pi : \mathcal{C} \to \mathcal{B}$ of smooth complex connected projective curves of genus g such that the monodromy representation of the Hitchin connection*

$$\rho_l : \pi_1(\mathcal{B}, b) \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{Z}_{l,b})$$

has an element of infinite order in its image.

For any genus $g \geq 2$ we give an example of a family $\pi : \mathcal{C} \to \mathcal{B}$ of smooth hyperelliptic curves of genus g and an explicit element $\xi \in \pi_1(\mathcal{B}, b)$ with image of infinite order (see Remark 6.10).

In the context of Witten-Reshetikhin-Turaev Topological Quantum Field Theory as defined by Blanchet-Habegger-Masbaum-Vogel [BHMV], the analogue of the above theorem is well-known due to work of Masbaum [Ma], who exhibited an explicit element of the mapping class group with image of infinite order. Previously, Funar [F] had shown by a different argument the somewhat weaker result that the image of the mapping class group is an infinite group.

It is enough to show the above theorem in the context of Conformal Field Theory as defined by Tsuchiya-Ueno-Yamada [TUY]: following a result of the first author [La], the monodromy representation associated to Hitchin's connection coincides with the monodromy representation of the WZW connection. In a series of papers by Andersen and Ueno ([AU1], [AU2], [AU3] and [AU4]) it has been shown recently that the above Conformal Field Theory and the above

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Topological Quantum Field Theory are equivalent. Therefore the above theorem also follows from that identification and the work of Funar and Masbaum.

In this short note, we give a direct algebraic proof, avoiding the above identification: we first recall Masbaum's initial argument applied to Tsuchiya-Kanie's description of the monodromy representation for the WZW connection in the case of the projective line with 4 marked points (see also [AMU]). Then we observe that the sewing procedure induces a projectively flat map between sheaves of conformal blocks, enabling us to increase the genus of the curve.

A couple of words about the exceptional levels $l = 1, 2, 4, 8$ are in order. For $l = 1$ the monodromy representation ρ_1 is finite for any g. This follows from the fact that the Beauville-Narasimhan-Ramanan [BNR] strange duality isomorphism $\mathbf{P}H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2)), \mathcal{L})^\dagger \xrightarrow{\sim} \mathbf{P}H^0(\mathrm{Pic}^{g-1}(\mathcal{C}_b), 2\Theta)$ is projectively flat over $\mathcal{B}$ for any family $\pi : \mathcal{C} \to \mathcal{B}$ (see e.g. [Be1]) and that ρ_1 thus identifies with the monodromy representation on a space of abelian theta functions, which is known to have finite image (see e.g. [W]). For $l = 2$ there is a canonical morphism $H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2)), \mathcal{L}^{\otimes 2}) \to H^0(\mathrm{Pic}^{g-1}(\mathcal{C}_b), 4\Theta)_+$, which is an isomorphism if and only if $\mathcal{C}_b$ has no vanishing theta-null [B]. But this map is not projectively flat having non-constant rank. So the question about finiteness of ρ_2 remains open — see also [Be2]. For $l = 4$ there is a canonical isomorphism [OP], [AM] between the dual $H^0(\mathrm{M}_{\mathcal{C}_b}(\mathrm{SL}(2)), \mathcal{L}^{\otimes 4})^\dagger$ and a space of abelian theta functions of order 3. We expect this isomorphism to be projectively flat. For $l = 8$ no isomorphism with spaces of abelian theta functions seems to be known.

Our motivation to study the monodromy representation of the Hitchin connection comes from the Grothendieck-Katz conjectures on the p-curvatures of a local system [K]. In a forthcoming paper we will discuss the consequences of the above theorem in this set-up.

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2. Review of mapping class groups, moduli spaces of pointed curves and braid groups

2.1. **Mapping class groups.** In this section we recall the basic definitions and properties of the mapping class groups. We refer the reader e.g. to [I] or [HL].

2.1.1. *Definitions.* Let S be a compact oriented surface of genus g without boundary and with n marked points $x_1, \ldots, x_n \in S$. Associated to the n-pointed surface S are the mapping class groups Γ_g^n and $\Gamma_{g,n}$ defined as the groups of isotopy classes of orientation-preserving diffeomorphisms $\phi : S \to S$ such that $\phi(x_i) = x_i$ for each i, respectively such that $\phi(x_i) = x_i$ and the differential $d\phi_{x_i} : T_{x_i}S \to T_{x_i}S$ at the point x_i is the identity map for each i.

An alternative definition of the mapping class groups Γ_g^n and $\Gamma_{g,n}$ can be given in terms of surfaces with boundary. We consider the surface R obtained from S by removing a small disc around each marked point x_i. The boundary ∂R consists of n circles. Equivalently, the groups Γ_g^n and $\Gamma_{g,n}$ coincide with the groups of isotopy classes of orientation-preserving diffeomorphisms $\phi : R \to R$ such that ϕ preserves each boundary component of R, respectively such that ϕ is the identity on ∂R.

The mapping class group Γ_g is defined to be $\Gamma_g^0 = \Gamma_{g,0}$.

2.1.2. *Dehn twists.* Given an (unparametrized) oriented, embedded circle γ in $R \subset S$ we can associate to it a diffeomorphism T_γ up to isotopy, i.e., an element T_γ in the mapping class groups Γ_g^n and $\Gamma_{g,n}$, the so-called Dehn twist along the curve γ. It is known that the mapping class groups Γ_g^n and $\Gamma_{g,n}$ are generated by a finite number of Dehn twists. We recall the following exact sequence

$$1 \longrightarrow \mathbf{Z}^n \longrightarrow \Gamma_{g,n} \longrightarrow \Gamma_g^n \longrightarrow 1.$$

The n generators of the abelian kernel $\mathbf{Z}^n$ are given by the Dehn twists T_{γ_i}, where γ_i is a loop going around the boundary circle associated to x_i for each i.

2.1.3. *The mapping class groups* Γ_0^4 *and* $\Gamma_{0,4}$. Because of their importance in this paper we recall the presentation of the mapping class groups Γ_0^4 and $\Gamma_{0,4}$ by generators and relations. Keeping the notation of the previous section, we denote by R the 4-holed sphere and by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ the circles in R around the four boundary circles. We denote by γ_{ij} the circle dividing R into two parts containing two holes each and such that the two circles γ_i and γ_j are in the same part. It is known (see e.g. [I] section 4) that $\Gamma_{0,4}$ is generated by the Dehn twists T_{γ_i} for $1 \leq i \leq 4$ and $T_{\gamma_{ij}}$ for $1 \leq i, j \leq 3$ and that, given a suitable orientation of the circles γ_i and γ_{ij}, there is a relation (the *lantern* relation)

$$T_{\gamma_1} T_{\gamma_2} T_{\gamma_3} T_{\gamma_4} = T_{\gamma_{12}} T_{\gamma_{13}} T_{\gamma_{23}}.$$

Note that the images of the Dehn twists T_{γ_i} under the natural homomorphism

$$\Gamma_{0,4} \longrightarrow \Gamma_0^4, \qquad T_\gamma \mapsto \overline{T}_\gamma,$$

are trivial. Thus the group Γ_0^4 is generated by the three Dehn twists $\overline{T}_{ij}$ for $1 \leq i, j \leq 3$ with the relation $\overline{T}_{\gamma_{12}} \overline{T}_{\gamma_{13}} \overline{T}_{\gamma_{23}} = 1$.

For each 4-holed sphere being contained in a closed genus g surface without boundary one can consider the Dehn twists T_{ij} as elements in the mapping class group Γ_g.

2.2. **Moduli spaces of curves.** Let $\mathfrak{M}_{g,n}$ denote the moduli space parameterizing n-pointed smooth projective curves of genus g. The moduli space $\mathfrak{M}_{g,n}$ is a (possibly singular) algebraic variety. It can also be thought of as an orbifold (or Deligne-Mumford stack) and one has an isomorphism

$$j : \pi_1(\mathfrak{M}_{g,n}, x) \xrightarrow{\sim} \Gamma_g^n, \tag{1}$$

where $\pi_1(\mathfrak{M}_{g,n}, x)$ stands for the orbifold fundamental group of $\mathfrak{M}_{g,n}$. In case the space $\mathfrak{M}_{g,n}$ is a smooth algebraic variety, the orbifold fundamental group coincides with the usual fundamental group.

2.3. **The isomorphism between** $\pi_1(\mathfrak{M}_{0,4}, x)$ **and** Γ_0^4**.** The moduli space $\mathfrak{M}_{0,4}$ parameterizes ordered sets of 4 points on the complex projective line $\mathbf{P}^1_{\mathbf{C}}$ up to the diagonal action of $\mathbf{PGL}(2, \mathbf{C})$. The cross-ratio induces an isomorphism with the projective line $\mathbf{P}^1_{\mathbf{C}}$ with 3 punctures at $0, 1$ and ∞

$$\mathfrak{M}_{0,4} \xrightarrow{\sim} \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}.$$

We deduce that the fundamental group of $\mathfrak{M}_{0,4}$ is the group with three generators

$$\pi_1(\mathfrak{M}_{0,4}, x) = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_3\sigma_2\sigma_1 = 1 \rangle,$$

where σ_1, σ_2 and σ_3 are the loops starting at $x \in \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\}$ and going once around the points $0, 1$ and ∞ with the same orientation. We choose the orientation such that the generators σ_i satisfy the relation $\sigma_3\sigma_2\sigma_1 = 1$. Clearly $\pi_1(\mathfrak{M}_{0,4}, x)$ coincides with the fundamental group $\pi_1(Q, x)$ of the 3-holed sphere Q.

In this particular case the isomorphism $j : \pi_1(\mathfrak{M}_{0,4}, x) \xrightarrow{\sim} \Gamma_0^4$ can be explicitly described as follows (see e.g. [I] Theorem 2.8.C): we may view the 3-holed sphere Q as the union of the 4-holed sphere R with a disc D glued on the boundary corresponding to the point x_4. Given a loop $\sigma \in \pi_1(Q, x)$ we may find an isotopy $\{f_t : Q \to Q\}_{0 \leq t \leq 1}$ such that the map $t \mapsto f_t(x)$ coincides with the loop σ, $f_0 = \mathrm{id}_Q$ and $f_1(D) = D$. Then the isotopy class of f_1 resticted to $R \subset Q$ determines an element $j(\sigma) = [f_1] \in \Gamma_0^4$. Moreover, with the previous notation, we have the equalities (see e.g. [I] Lemma 4.1.I)

$$j(\sigma_1) = \overline{T}_{\gamma_{23}}, \qquad j(\sigma_2) = \overline{T}_{\gamma_{13}}, \qquad j(\sigma_3) = \overline{T}_{\gamma_{12}}.$$

Remark 2.1. At this stage we observe that under the isomorphism j the two elements $\sigma_1^{-1}\sigma_2 \in \pi_1(\mathfrak{M}_{0,4}, x)$ and $\overline{T}_{\gamma_{23}}^{-1}\overline{T}_{\gamma_{13}} \in \Gamma_0^4$ coincide. It was shown by G. Masbaum in [Ma] that the latter element has infinite order in the TQFT-representation of the mapping class group Γ_g — note that $T_{\gamma_{23}}^{-1}T_{\gamma_{13}}$ also makes sense in Γ_g. We will show in Proposition 5.1 that the loop $\sigma_1^{-1}\sigma_2$ has infinite order in the monodromy representation of the WZW connection.

2.4. **Braid groups and configuration spaces.** We recall some basic results about braid groups and configuration spaces. We refer the reader e.g. to [KT] Chapter 1.

2.4.1. *Definitions.* The braid group B_n is the group generated by $n-1$ generators $g_1, \ldots, g_{n-1}$ and the relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \; 1 \leq i \leq n-2, \qquad \text{and} \qquad g_i g_j = g_j g_i, \; |i-j| \geq 2.$$

The pure braid group is the kernel $P_n = \ker(B_n \to \Sigma_n)$ of the group homomorphism which associates to the generator g_i the transposition $(i, i+1)$ in the symmetric group Σ_n. The braid groups B_n and P_n can be identified with the fundamental groups

$$P_n = \pi_1(X_n, p), \qquad B_n = \pi_1(\overline{X}_n, \overline{p}),$$

where X_n and $\overline{X}_n$ are the complex manifolds parameterizing ordered respectively unordered n-tuples of distinct points in the complex plane

$$X_n = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n \mid z_i \neq z_j\} \qquad \text{and} \qquad \overline{X}_n = X_n/\Sigma_n.$$

The points $p = (z_1, \ldots, z_n)$ and $\overline{p} = p \bmod \Sigma_n$ are base points in X_n and $\overline{X}_n$. There are natural inclusions $B_n \hookrightarrow B_{n+1}$, which induce inclusions on the pure braid groups $\iota : P_n \hookrightarrow P_{n+1}$.

Over the variety X_n there is an universal family

$$\mathcal{F}_{n+1} = (\pi : \mathcal{C} = X_n \times \mathbf{P}^1 \to X_n; s_1, \ldots, s_n, s_\infty), \tag{2}$$

parameterizing $n+1$ distinct points on the projective line $\mathbf{P}^1$. The section s_i is given by the natural projection $X_n \to \mathbf{C}$ on the i-th component followed by the inclusion $\mathbf{C} \subset \mathbf{P}^1_{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and s_∞ is the constant section corresponding to $\infty \in \mathbf{P}^1_{\mathbf{C}}$.

2.4.2. *Relation between the pure braid group P_3 and the fundamental group $\pi_1(\mathfrak{M}_{0,4}, x)$.* The natural map

$$\mathfrak{M}_{0,4} = \mathbf{P}^1_{\mathbf{C}} \setminus \{0, 1, \infty\} \longrightarrow X_3, \qquad z \mapsto (0, 1, z)$$

induces a group homomorphism at the level of fundamental groups

$$\Psi : \pi_1(\mathfrak{M}_{0,4}, x) = \langle \sigma_1, \sigma_2 \rangle \longrightarrow P_3 = \pi_1(X_3, p_3),$$

with $p_3 = (0, 1, x)$. Then Ψ is a monomorphism by [KT] Theorem 1.16. Moreover, the image of Ψ coincides with the kernel of the natural group homomorphism

$$\mathrm{im}\ \Psi = \ker\left(P_3 = \pi_1(X_3, p_3) \longrightarrow P_2 = \pi_1(X_2, p_2)\right)$$

induced by the projection onto the first two factors $X_3 \to X_2$, $(z_1, z_2, z_3) \mapsto (z_1, z_2)$ and $p_2 = (0, 1)$. One computes explicitly (see [KT] section 1.4.2) that

$$\Psi(\sigma_1) = g_2 g_1^2 g_2^{-1}, \qquad \text{and} \qquad \Psi(\sigma_2) = g_2^2.$$

For later use we introduce the element

$$\sigma = \sigma_1^{-1}\sigma_2 \in \pi_1(\mathfrak{M}_{0,4}, x). \tag{3}$$

3. Conformal blocks and the projective WZW connection

3.1. **General set-up.** We consider the simple Lie algebra $\mathfrak{sl}(2)$. The set of irreducible $\mathfrak{sl}(2)$-modules, i.e. the set of dominant weights of $\mathfrak{sl}(2)$ equals

$$P_+ = \{\lambda = m\varpi \mid m \in \mathbf{N}\},$$

where ϖ is the fundamental weight of $\mathfrak{sl}(2)$, which corresponds to the standard 2-dimensional representation of $\mathfrak{sl}(2)$. We fix an integer $l \geq 1$, called the level, and introduce the finite set $P_l = \{\lambda \in P_+ \mid m \leq l\}$. Given any $\lambda \in P_l$ we denote by $\lambda^\dagger \in P_l$ the dominant weight of the dual $V_\lambda^\dagger$ of the $\mathfrak{sl}(2)$-module V_λ with dominant weight λ. Note that $\lambda^\dagger = \lambda$. Given an integer $n \geq 1$, a collection $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (P_l)^n$ of dominants weights of $\mathfrak{sl}(2)$ and a family

$$\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_n; \xi_1, \ldots, \xi_n)$$

of n-pointed stable curves of arithmetic genus g parameterized by a base variety $\mathcal{B}$ with sections $s_i : \mathcal{B} \to \mathcal{C}$ and formal coordinates ξ_i at the divisor $s_i(\mathcal{B}) \subset \mathcal{C}$, one constructs (see [TUY] section 4.1) a locally free sheaf

$$\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$$

over the base variety $\mathcal{B}$, called the *sheaf of conformal blocks* or the *sheaf of vacua*. We recall that $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ is a subsheaf of $\mathcal{O}_\mathcal{B} \otimes \mathcal{H}^\dagger_{\vec{\lambda}}$, where $\mathcal{H}^\dagger_{\vec{\lambda}}$ denotes the dual of the tensor product $\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$ of the integrable highest weight representations $\mathcal{H}_{\lambda_i}$ of level l and weight λ_i of the affine Lie algebra $\widehat{\mathfrak{sl}(2)}$. The formation of the sheaf of conformal blocks commutes with base change. In particular, we have for any point $b \in \mathcal{B}$

$$\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_b \cong \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_b),$$

where $\mathcal{F}_b$ denotes the data $(\mathcal{C}_b = \pi^{-1}(b); s_1(b), \ldots, s_n(b); \xi_{1|\mathcal{C}_b}, \ldots, \xi_{n|\mathcal{C}_b})$ consisting of a stable curve $\mathcal{C}_b$ with n marked points $s_1(b), \ldots, s_n(b)$ and formal coordinates $\xi_{i|\mathcal{C}_b}$ at the points $s_i(b)$.

We recall that the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ does not depend (up to a canonical isomorphism) on the formal coordinates ξ_i (see e.g. [U] Theorem 4.1.7). We therefore omit the formal coordinates in the notation.

3.2. **The projective WZW connection.** We now outline the definition of the projective WZW connection on the sheaf $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ over the smooth locus $\mathcal{B}^s \subset \mathcal{B}$ parameterizing smooth curves and refer to [TUY] or [U] for a detailed account. Let $\mathcal{D} \subset \mathcal{B}$ be the discriminant locus and let $\mathcal{S} = \coprod_{i=1}^n s_i(\mathcal{B})$ be the union of the images of the n sections. We recall the exact sequence

$$0 \longrightarrow \pi_* \Theta_{\mathcal{C}/\mathcal{B}}(*\mathcal{S}) \longrightarrow \pi_* \Theta'_{\mathcal{C}}(*\mathcal{S})_\pi \xrightarrow{\theta} \Theta_{\mathcal{B}}(-\log \mathcal{D}) \longrightarrow 0, \tag{4}$$

where $\Theta_{\mathcal{C}/\mathcal{B}}(*\mathcal{S})$ denotes the sheaf of vertical rational vector fields on $\mathcal{C}$ with poles only along the divisor $\mathcal{S}$, and $\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi$ the sheaf of rational vector fields on $\mathcal{C}$ with poles only along the divisor $\mathcal{S}$ and with constant horizontal components along the fibers of π. There is an $\mathcal{O}_{\mathcal{B}}$-linear map

$$p : \pi_* \Theta'_{\mathcal{C}}(*\mathcal{S})_\pi \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathcal{B}}((\xi_i)) \frac{d}{d\xi_i},$$

which associates to a vector field $\vec{\ell}$ in $\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi$ the n Laurent expansions $\ell_i \frac{d}{d\xi_i}$ around the divisor $s_i(\mathcal{B})$. Abusing notation we also write $\vec{\ell}$ for its image under p

$$\vec{\ell} = (\ell_1 \frac{d}{d\xi_1}, \cdots, \ell_n \frac{d}{d\xi_n}) \in \bigoplus_{i=1}^n \mathcal{O}_{\mathcal{B}}((\xi_i)) \frac{d}{d\xi_i}.$$

We then define for any vector field $\vec{\ell}$ in $\Theta'_{\mathcal{C}}(*\mathcal{S})_\pi$ the endomorphism $D(\vec{\ell})$ of $\mathcal{O}_{\mathcal{B}} \otimes \mathcal{H}^\dagger_{\vec{\lambda}}$ by

$$D(\vec{\ell})(f \otimes u) = \theta(\vec{\ell}).f \otimes u + \sum_{i=1}^n f \otimes (T[\ell_i].u)$$

for f a local section of $\mathcal{O}_{\mathcal{B}}$ and $u \in \mathcal{H}^\dagger_{\vec{\lambda}}$. Here $T[\ell_i]$ denotes the action of the energy-momentum tensor on the i-th component $\mathcal{H}^\dagger_{\lambda_i}$. It is shown in [TUY] that $D(\vec{\ell})$ preserves $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ and that $D(\vec{\ell})$ only depends on the image $\theta(\vec{\ell})$ up to homothety. One therefore obtains a projective connection ∇ on the sheaf $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ over $\mathcal{B}^s$ given by

$$\nabla_{\theta(\vec{\ell})} = \theta(\vec{\ell}) + T[\vec{\ell}].$$

Since this connection is projectively flat, it induces a monodromy representation

$$\rho_{l,\vec{\lambda}} : \pi_1(\mathcal{B}^s, b) \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})_b)$$

for some base point $b \in \mathcal{B}^s$.

Remark 3.1. For a family of smooth n-pointed curves of genus 0 the projective WZW connection is actually a connection (see e.g. [U] section 5.4).

4. Monodromy of the WZW connection for a family of 4-pointed rational curves

In this section we review the results by Tsuchiya-Kanie [TK] on the monodromy of the WZW connection for a family of rational curves with 4 marked points. We consider the universal family $\mathcal{F}_4$ over X_3 introduced in (2) with the collection

$$\vec{\lambda}^{TK} = (\varpi, \varpi, \varpi, \varpi) \in (P_l)^4.$$

The rank of the sheaf of conformal blocks $\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4)$ equals 2 for any $l \geq 1$, see e.g. [TK] Theorem 3.3. Moreover, as outlined in section 3.2, the bundle $\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4)$ is equipped with a flat connection ∇ (not only projective).

Remark 4.1. It is known [TK] that the differential equations satisfied by the flat sections of $(\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4), \nabla)$ coincide with the Knizhnik-Zamolodchikov equations (see e.g. [EFK]). Moreover, we will show in a forthcoming paper that the local system $(\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4), \nabla)$ also coincides with a certain Gauss-Manin local system.

We observe that the symmetric group Σ_3 acts naturally on the base variety X_3. The local system $(\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4), \nabla)$ is invariant under this Σ_3-action and admits a natural Σ_3-linearization. Thus by descent we obtain a local system $(\overline{\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4)}, \overline{\nabla})$ over $\overline{X}_3$. Therefore, we obtain a monodromy representation

$$\widetilde{\rho}_l : B_3 = \pi_1(\overline{X}_3, \overline{p}) \longrightarrow \mathrm{GL}(\overline{\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4)}_{\overline{p}}) = \mathrm{GL}(2, \mathbf{C})$$

Proposition 4.2 ([TK] Theorem 5.2)**.** *We put $q = \exp(\frac{2i\pi}{l+2})$. There exists a basis B of the vector space $\overline{\mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4)}_{\overline{p}} = \mathcal{V}^{\dagger}_{l,\vec{\lambda}^{TK}}(\mathcal{F}_4)_p$ such that*

$$\mathrm{Mat}_B(\widetilde{\rho}_l(g_1)) = q^{-\frac{3}{4}} \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{Mat}_B(\widetilde{\rho}_l(g_2)) = \frac{q^{-\frac{3}{4}}}{q+1} \begin{pmatrix} -1 & t \\ t & q^2 \end{pmatrix},$$

with $t = \sqrt{q(1+q+q^2)}$. Note that both matrices have eigenvalues $q^{\frac{1}{4}}$ and $-q^{-\frac{3}{4}}$.

Remark 4.3. These matrices have already been used in the paper [AMU].

5. Infinite monodromy over $\mathfrak{M}_{0,4}$

We denote by ρ_l the restriction of the monodromy representation $\widetilde{\rho}_l$ to the subgroup $\pi_1(\mathfrak{M}_{0,4}, x)$ of B_3 (see section 2.4.2)

$$\rho_l : \pi_1(\mathfrak{M}_{0,4}, x) \subset B_3 \longrightarrow \mathrm{GL}(2, \mathbf{C}).$$

Proposition 5.1. *Let $\sigma \in \pi_1(\mathfrak{M}_{0,4}, x)$ be the element introduced in* (3)*. If $l \neq 1, 2, 4$ and 8, then the element $\rho_l(\sigma)$ has infinite order in both $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$ and $\mathrm{GL}(2, \mathbf{C})$*

Proof. Using the explicit form of the monodromy representation ρ_l given in Proposition 4.2 we compute the matrix associated to $\Psi(\sigma) = \Psi(\sigma_1^{-1}\sigma_2) = g_2 g_1^{-2} g_2$

$$\mathrm{Mat}_B(\widetilde{\rho}_l(\Psi(\sigma))) = \frac{1}{(q+1)^2} \begin{pmatrix} q^{-2} + t^2 & t(q^2 - q^{-2}) \\ t(q^2 - q^{-2}) & t^2 q^{-2} + q^4 \end{pmatrix}.$$

This matrix has determinant 1 and trace $2 - q - q^{-1} + q^2 + q^{-2}$. Hence the matrix has finite order if and only if there exists a primitive root of unity λ such that

$$\lambda + \lambda^{-1} = 2 - q - q^{-1} + q^2 + q^{-2}.$$

In [Ma] it is shown that this can only happen if $l = 1, 2, 4$ or 8: using the transitive action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on primitive roots of unity, one gets that, if such a λ exists for $q = \exp(\frac{2i\pi}{l+2})$, then for *any* primitive $(l+2)$-th root $\tilde{q}$ there exists a primitive root $\tilde{\lambda}$ such that

$$\tilde{\lambda} + \tilde{\lambda}^{-1} = 2 - \tilde{q} - \tilde{q}^{-1} + \tilde{q}^2 + \tilde{q}^{-2}.$$

In particular, we have the inequality $|1 - \mathbf{R}\mathrm{e}(\tilde{q}) + \mathbf{R}\mathrm{e}(\tilde{q}^2)| \leq 1$ for any primitive $(l+2)$-th root $\tilde{q}$. But for $l \neq 1, 2, 4$ and 8, one can always find a primitive $(l+2)$-th root $\tilde{q}$ such that $\mathbf{R}\mathrm{e}(\tilde{q}^2) > \mathbf{R}\mathrm{e}(\tilde{q})$ — for the explicit root $\tilde{q}$ see [Ma].

Finally, since $\rho_l(\sigma)$ has trivial determinant, its class in $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$ will also have infinite order. $\square$

Remark 5.2. The same computation shows that the element $\rho_l(\sigma_1\sigma_2^{-1}) \in \mathrm{GL}(2, \mathbf{C})$ also has infinite order if $l \neq 1, 2, 4$ and 8. This implies that the orientation chosen for both loops σ_1 and σ_2 around 0 and 1 is irrelevant. On the other hand, it is immediately seen that the elements $\rho_l(\sigma_1), \rho_l(\sigma_2)$ and $\rho_l(\sigma_1\sigma_2)$ have finite order for any level l.

Proposition 5.3. *In the four cases $l = 1, 2, 4$ and 8, the image $\rho_l(\pi_1(\mathfrak{M}_{0,4}, x))$ in the projective linear group $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$ is finite and isomorphic to the groups given in table*

l	1	2	4	8
$\rho_l(\pi_1(\mathfrak{M}_{0,4}, x))$	μ_3	$\mu_2 \times \mu_2$	A_4	A_5

Here A_n denotes the alternating group on n letters.

Proof. We denote by $m_1, m_2 \in \mathbf{P}\mathrm{GL}(2, \mathbf{C})$ the elements defined by the matrices $\mathrm{Mat}_B(\rho_l(\sigma_1))$ and $\mathrm{Mat}_B(\rho_l(\sigma_2))$ and denote by $\mathrm{ord}(m_i)$ their order in the group $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$. In the first two cases one immediately checks the relations $m_1 = m_2$, $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = 3$ (for $l = 1$) and $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = \mathrm{ord}(m_1m_2) = 2$ (for $l = 2$).

In the case $l = 4$ we recall that the alternating group A_4 has the following presentation by generators and relations

$$A_4 = \langle a, b \mid a^3 = b^2 = (ab)^3 = 1 \rangle.$$

Using the formulae of Proposition 4.2 and 5.1 we check that $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = 3$ and $\mathrm{ord}(m_1^{-1}m_2) = 2$, so that $a = m_1$ and $b = m_1^{-1}m_2$ generate the group A_4.

In the case $l = 8$ we recall that the alternating group A_5 has the following presentation by generators and relations

$$A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle.$$

Using the formulae of Proposition 4.2 and 5.1 we check that $\mathrm{ord}(m_1) = \mathrm{ord}(m_2) = 5$ and $\mathrm{ord}(m_1^{-1}m_2) = 3$. Moreover a straightforward computation shows that the element $m_1^{-1}m_2m_1^{-1}$ is (up to a scalar) conjugate to the matrix

$$\mathrm{Mat}_B(\widetilde{\rho}_l(g_1^{-2}g_2^2g_1^{-2})) = * \begin{pmatrix} q^{-4}(1+t^2) & t(1-q^{-2}) \\ t(1-q^{-2}) & t^2+q^4 \end{pmatrix},$$

which has trace zero. Note that $t^2 = q + q^2 + q^3$ and $q^{-4} = -q$. Hence $\mathrm{ord}(m_1^{-1}m_2m_1^{-1}) = \mathrm{ord}(m_1m_2^{-1}m_1) = 2$. Therefore if we put $a = m_1m_2^{-1}m_1$ and $b = m_1^{-1}m_2$, we have $ab = m_1$ and $ab^2 = m_2$, so that $\mathrm{ord}(a) = 2$, $\mathrm{ord}(b) = 3$, and $\mathrm{ord}(ab) = 5$, i.e. a, b generate the group A_5. $\square$

Corollary 5.4. *In the four cases $l = 1, 2, 4$ and 8, the image $\widetilde{\rho}_l(B_3)$ in $\mathrm{GL}(2, \mathbf{C})$ is finite.*

Proof. First, we observe that the image $\rho_l(\pi_1(\mathfrak{M}_{0,4}, x))$ in $\mathrm{GL}(2, \mathbf{C})$ is finite. In fact, by Proposition 5.3 its image in $\mathbf{P}\mathrm{GL}(2, \mathbf{C})$ is finite and its intersection $\rho_l(\pi_1(\mathfrak{M}_{0,4}, x)) \cap \mathbf{C}^*\mathrm{Id}$ with the center of $\mathrm{GL}(2, \mathbf{C})$ is also finite. The latter follows from the fact that the determinant $\det \mathrm{Mat}_B(\widetilde{\rho}_l(g_i)) = -q^{-\frac{1}{2}}$ has finite order in $\mathbf{C}^*$.

Secondly, we recall that P_3 is generated by the normal subgroup $\pi_1(\mathfrak{M}_{0,4}, x)$ and by the element g_1^2. Since $\widetilde{\rho}_l(g_1^2)$ has finite order and since $B_3/P_3 = \Sigma_3$ is finite, we obtain that $\widetilde{\rho}_l(B_3)$ is a finite subgroup. □

In the proof of the main theorem we will need the following corollary of Proposition 5.1. We consider the following compact subset of $\mathbf{C}$

$$\mathcal{B} = D \setminus (\Delta_{-1} \cup \Delta_0 \cup \Delta_1) \tag{5}$$

where $D \subset \mathbf{C}$ is the closed disc centered at 0 with radius 2 and $\Delta_z \subset \mathbf{C}$ denotes the open disc centered at z with very small radius. We choose as base point $b = i \in \mathcal{B}$. Let $\xi \in \pi_1(\mathcal{B}, b)$ be the loop going once around the two points -1 and 0. Let

$$\mathcal{F} = (\mathbf{P}^1 \times \mathcal{B} \to \mathcal{B}; s_0, s_1, s_{-1}, s_u, s_{-u})$$

be the family of 5-pointed rational curves, where the 5 sections $s_0, s_1, s_{-1}, s_u, s_{-u}$ map $u \in \mathcal{B}$ to $0, 1, -1, u$ and $-u$ respectively.

Corollary 5.5. *For $l \neq 1, 2, 4$ and 8, the image of the loop $\xi \in \pi_1(\mathcal{B}, b)$ under the monodromy representation*

$$\rho_l : \pi_1(\mathcal{B}, b) \longrightarrow \mathrm{GL}(\mathcal{V}^\dagger_{l,0,\vec{\lambda}^{TK}}(\mathcal{F}))$$

has infinite order.

Proof. Since propagation of vacua is a flat isomorphism (see e.g. [Lo] Proposition 22) we can drop the point 0 which is marked with the zero weight. Thus it suffices to show the statement for the same family with the 4 sections s_1, s_{-1}, s_u, s_{-u}. The cross-ratio of the 4 points $1, -1, u, -u$ equals

$$t = \frac{(u+1)^2}{4u} = \frac{1}{2} + \frac{1}{4}(u + u^{-1}).$$

We also introduce the 4-pointed family

$$\mathcal{F}' = (\mathbf{P}^1 \times \mathcal{B} \to \mathcal{B}; s_0, s_1, s_\infty, s_t),$$

where the section s_t maps u to the cross-ratio t and we observe that there exists an automorphism $\alpha : \mathbf{P}^1 \times \mathcal{B} \to \mathbf{P}^1 \times \mathcal{B}$ over $\mathcal{B}$ (which can be made explicit) mapping the 4 sections s_1, s_{-1}, s_u, s_{-u} to the 4 sections s_0, s_1, s_∞, s_t. Moreover the automorphism α induces an isomorphism between the two local systems $(\mathcal{V}^\dagger_{l,\vec{\lambda}^{TK}}(\mathcal{F}), \nabla)$ and $(\mathcal{V}^\dagger_{l,\vec{\lambda}^{TK}}(\mathcal{F}'), \nabla)$ over $\mathcal{B}$. We now consider the map induced by the cross-ratio

$$\Psi : \mathcal{B} \longrightarrow \mathbf{P}^1 \setminus \{0, 1, \infty\} = \mathfrak{M}_{0,4}, \qquad u \mapsto t.$$

One easily checks that the extension $\overline{\Psi}$ of Ψ to $\mathbf{P}^1$ gives a double cover of $\mathbf{P}^1$ ramified over $0 = \overline{\Psi}(-1)$ and $1 = \overline{\Psi}(1)$. Note that $\overline{\Psi}(0) = \overline{\Psi}(\infty) = \infty$. Hence Ψ is an étale double cover over its image. The map Ψ induces a map, denoted by Φ, between fundamental groups

$$\Phi : \pi_1(\mathcal{B}, i) \longrightarrow \pi_1(\mathbf{P}^1 \setminus \{0, 1, \infty\}, \frac{1}{2}).$$

An elementary computations shows that $\Phi(\xi_1) = \sigma_1^2$, $\Phi(\xi_2) = \sigma_2^2$, and $\Phi(\xi_3) = \sigma_2^{-1}\sigma_1^{-1}$, where ξ_1, ξ_2 and ξ_3 denote the loops in $\mathcal{B}$ going once around the points -1, 1 and 0 respectively. We recall from section 2.3 that σ_1 and σ_2 denote the loops in $\mathfrak{M}_{0,4}$ around the points 0 and 1. All orientations of the loops are the same. Hence if we denote $\xi = \xi_3\xi_1$, the loop in $\mathcal{B}$ going around the two points -1 and 0, then $\Phi(\xi) = \sigma_2^{-1}\sigma_1 = \sigma^{-1}$. Thus ξ has infinite order by Proposition 5.1. □

6. Infinite monodromy for higher genus

6.1. **Deformation of families of pointed nodal curves.** In this section we explicitly describe a deformation of two families of rational nodal curves into smooth curves. These deformations will be used in the proof of the main theorem.

6.1.1. *A family of rational curves.* We consider the family of rational curves $p : \mathcal{C} \to \mathbf{A}^1$ parameterized by the affine line $\mathbf{A}^1$ and given by the equation

$$\mathcal{C} = \mathrm{Zeros}(f) \subset \mathbf{P}^2 \times \mathbf{A}^1 \qquad f = xy - \tau z^2,$$

where $(x : y : z)$ are homogeneous coordinates on the projective plane and τ is a coordinate on $\mathbf{A}^1$. We denote by $\mathcal{C}_\tau$ the fiber over $\tau \in \mathbf{A}^1$. For $\tau \neq 0$ the curve $\mathcal{C}_\tau$ is a smooth conic and $\mathcal{C}_0 = L_0 \cup L_1$ is the union of two projective lines given by the equations $L_0 = \mathrm{Zeros}(y)$ and $L_1 = \mathrm{Zeros}(x)$. For $\tau \neq 0$ we can parameterize the smooth conic $\mathcal{C}_\tau$ in the following way

$$\Phi_\tau : \mathbf{P}^1 \longrightarrow \mathcal{C}_\tau \subset \mathbf{P}^2, \qquad (\alpha : \beta) \mapsto (\beta^2 : \tau\alpha^2 : \alpha\beta).$$

Note that for $\tau = 0$ this morphism also gives a parametrization of the line L_0.

Let $m \geq 2$ be an integer. We define $2m+1$ sections $s_1, \ldots, s_{2m}, s_\infty$ of the family $p : \mathcal{C} \to \mathbf{A}^1$ parameterized by an open subset of $\mathbf{A}^1$ with coordinate u. We put

$$s_1(u,\tau) = (1 : \tau : 1) \qquad s_2(u,\tau) = (1 : \tau : -1),$$

$$s_3(u,\tau) = (1 : \tau u^2 : u) \qquad s_4(u,\tau) = (1 : \tau u^2 : -u),$$

and for $j = 3, \ldots, m$

$$s_{2j-1}(u,\tau) = (\tau : j^2 : j) \qquad s_{2j}(u,\tau) = (\tau : j^2 : -j).$$

Finally we put

$$s_\infty(u,\tau) = (0 : 1 : 0).$$

We observe that for $\tau \neq 0$ the $2m+1$ points $s_1(u,\tau), \ldots, s_{2m}(u,\tau), s_\infty(u,\tau)$ correspond to the following points in $\mathbf{P}^1$ via the morphism Φ_τ:

$$1, -1, u, -u, 3\tau^{-1}, -3\tau^{-1}, \ldots, m\tau^{-1}, -m\tau^{-1}, \infty.$$

For $\tau = 0$ the points $s_1(u,0), s_2(u,0), s_3(u,0), s_4(u,0) \in L_0$ have coordinates $1, -1, u, -u$ and the points $s_5(u,0), \ldots, s_{2m}(u,0), s_\infty(u,0) \in L_1$ have coordinates $3, -3, \ldots, m, -m, \infty$, in particular they do not depend on u. We consider the open subset $\Omega = \{\tau \ : \ |\tau| < \frac{1}{2}\} \subset \mathbf{A}^1$ and the compact subset $\mathcal{B}$ as defined in (5), and define the family

$$\mathcal{F}^{rat}_{2m+1} = (\pi = id \times p : \mathcal{B} \times \mathcal{C}_{|\Omega} \to \mathcal{B} \times \Omega; s_1, s_2, \ldots, s_{2m}, s_\infty)$$

of $2m+1$-pointed rational curves.

6.1.2. *A family of hyperelliptic curves.* Let $g \geq 2$ be an integer and let α be a complex number satisfying $|\alpha| > 1$. We consider the family $p : \mathcal{C} \to \mathbf{A}^1 \times \mathbf{A}^1$ of curves parameterized by two complex numbers $(u,\tau) \in \mathbf{A}^1 \times \mathbf{A}^1$ and such that the fiber $\mathcal{C}_{(u,\tau)}$ is the double cover of $\mathbf{P}^1$ ramified over the $2g+2$ points:

$$0, \infty, u^2 + \tau, u^2 - \tau, 1 + \tau, 1 - \tau, (3\alpha)^2 + \tau, (3\alpha)^2 - \tau, \ldots, (g\alpha)^2 + \tau, (g\alpha)^2 - \tau.$$

We assume that these points are distinct. We denote the projection $pr : \mathcal{C}_{(u,\tau)} \to \mathbf{P}^1$. The family of curves $\mathcal{C}$ can be constructed by taking the closure in $\mathbf{P}^2 \times \mathbf{A}^1 \times \mathbf{A}^1$ of the affine curve in $\mathbf{A}^2 \times \mathbf{A}^1 \times \mathbf{A}^1$ over $\mathbf{A}^1 \times \mathbf{A}^1$ defined by the equation

$$(6) \quad y^2 = x(x-1+\tau)(x-1-\tau)(x-u^2+\tau)(x-u^2-\tau)\prod_{j=3}^{g}(x-(j\alpha)^2+\tau)(x-(j\alpha)^2-\tau)$$

and by blowing up g times the singular point at ∞.

We notice that for any $\tau \neq 0$ with $|\tau|$ sufficiently small and for u varying in a Zariski open subset U_τ of $\mathbf{A}^1$ the curves $\mathcal{C}_{(u,\tau)}$ are smooth hyperelliptic curves of genus g. For $\tau = 0$ and $u^2 \neq 0, 1, (3\alpha)^2, \ldots, (g\alpha)^2$ the curve $\mathcal{C}_{(u,\tau)}$ is a rational nodal curve with g nodes lying over the points $1, u^2, (3\alpha)^2, \ldots, (g\alpha)^2$ of $\mathbf{P}^1$. The normalization map $\eta : \mathbf{P}^1 \to \mathcal{C}_{(u,0)}$ is explicitly given over $\mathbf{A}^1 \subset \mathbf{P}^1$ by the expressions

$$\eta : \mathbf{A}^1 \to \mathbf{A}^2, \qquad t \mapsto (x(t), y(t)) = (t^2, t(t^2-1)(t^2-u^2)\prod_{j=3}^{g}(t^2-(j\alpha)^2).$$

This shows that the pre-images by η of the g nodes are

$$1, -1, u, -u, 3\alpha, -3\alpha, \ldots, g\alpha, -g\alpha.$$

We put $s_\infty(u,\tau) = \infty \in \mathcal{C}_{(u,\tau)}$ for any (u,τ) and $\mathcal{B}$ as defined in (5). Let $\Omega = \{\tau : |\tau| < \frac{1}{2}\}$. We define the family

$$\mathcal{F}_g^{hyp} := (\pi : \mathcal{C}_{|\mathcal{B}\times\Omega} \to \mathcal{B} \times \Omega, s_\infty)$$

of 1-pointed hyperelliptic curves.

6.2. **The sewing procedure.** We will briefly sketch the construction of the sewing map and give some of its properties (for the details see [TUY] or [U]).

We consider a flat family

$$\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B} \times \Omega; s_1, \ldots, s_n)$$

of n-pointed connected projective curves parameterized by $\mathcal{B}\times\Omega$, where $\mathcal{B}$ is a complex manifold and $\Omega \subset \mathbf{A}^1$ is an open subset of the complex affine line $\mathbf{A}^1$ containing the origin 0. We assume that the family $\mathcal{F}$ satisfies the following conditions:

(1) the curve $\mathcal{C}_{(b,\tau)}$ is smooth if $\tau \neq 0$.
(2) the curve $\mathcal{C}_{(b,0)}$ has exactly one node.

We also introduce the family $\widetilde{\mathcal{F}}$ of $n+2$-pointed curves associated to $\mathcal{F}$

$$\widetilde{\mathcal{F}} = (\widetilde{\pi} : \widetilde{\mathcal{C}} \to \mathcal{B}; s_1, \ldots, s_{n+2})$$

which desingularizes the family of nodal curves $\mathcal{F}_{|\mathcal{B}\times\{0\}}$. Here $s_{n+1}(b)$ and $s_{n+2}(b)$ are the two points of $\widetilde{\mathcal{C}}_b$ lying over the node of $\mathcal{C}_{(b,0)}$.

Remark 6.1. An example of a family $\mathcal{F}$ satisfying the above conditions is given in section 6.1.1.

For any dominant weight μ the Virasoro operator L_0 induces a decomposition of the representation space $\mathcal{H}_\mu$ into a direct sum of eigenspaces $\mathcal{H}_\mu(d)$ for the eigenvalue $d+\Delta_\mu$ of L_0, where $\Delta_\mu \in \mathbf{Q}$ is the trace anomaly and $d \in \mathbf{N}$. We recall that there exists a unique (up to a scalar) bilinear pairing

$$(.|.) : \mathcal{H}_\mu \times \mathcal{H}_{\mu^\dagger} \to \mathbf{C} \qquad \text{such that} \qquad (X(n)u|v) + (u|X(-n)v) = 0$$

for any $X \in \mathfrak{sl}(2)$, $n \in \mathbf{Z}$, $u \in \mathcal{H}_\mu$, $v \in \mathcal{H}_{\mu^\dagger}$ and $(.|.)$ is zero on $\mathcal{H}_\mu(d) \times \mathcal{H}_{\mu^\dagger}(d')$ if $d \neq d'$. We choose a basis $\{v_1(d), \ldots, v_{m_d}(d)\}$ of $\mathcal{H}_\mu(d)$ and let $\{v^1(d), \ldots, v^{m_d}(d)\}$ be its dual basis of $\mathcal{H}_{\mu^\dagger}(d)$ with respect to the above bilinear form. Then the element

$$\gamma_d = \sum_{i=1}^{m_d} v_i(d) \otimes v^i(d) \in \mathcal{H}_\mu(d) \otimes \mathcal{H}_{\mu^\dagger}(d) \subset \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^\dagger}$$

does not depend on the basis. We recall that $\mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})$ is a locally free $\mathcal{O}_\mathcal{B}$-module. Given a section $\psi \in \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})$ we define an $\mathcal{H}^\dagger_{\vec{\lambda}}$-valued power series $\widetilde{\psi} \in \mathcal{H}^\dagger_{\vec{\lambda}}[[\tau]] \otimes \mathcal{O}_\mathcal{B}$ as follows. For any non-negative integer d the inclusion $\mathcal{H}_{\vec{\lambda}} \hookrightarrow \mathcal{H}_{\vec{\lambda},\mu,\mu^\dagger}$, $v \mapsto v \otimes \gamma(d)$ induces a dual projection

$$\pi_d : \mathcal{H}^\dagger_{\vec{\lambda},\mu,\mu^\dagger} \longrightarrow \mathcal{H}^\dagger_{\vec{\lambda}}. \tag{7}$$

We denote by ψ_d the image $\pi_d(\psi) \in \mathcal{H}^\dagger_{\vec{\lambda}} \otimes \mathcal{O}_\mathcal{B}$. We then define

$$\widetilde{\psi} = \sum_{d=0}^{\infty} \psi_d \tau^d \in \mathcal{H}_{\vec{\lambda}}[[\tau]] \otimes \mathcal{O}_\mathcal{B}.$$

It is shown in [TUY] that

$$\widetilde{\psi} \in \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{B}\times\Omega}} \widehat{\mathcal{O}},$$

where $\widehat{\mathcal{O}}$ denotes the structure sheaf of the completion of $\mathcal{B} \times \Omega$ along the divisor $\mathcal{B} \times \{0\}$. Note that $\widehat{\mathcal{O}} = \mathcal{O}_\mathcal{B}[[\tau]]$. Therefore we obtain for any $\mu \in P_l$ and any $\vec{\lambda} \in (P_l)^n$ an $\mathcal{O}_{\mathcal{B}\times\Omega}$-linear map

$$s_\mu : \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}}) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_{\mathcal{B}\times\Omega} \longrightarrow \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{B}\times\Omega}} \widehat{\mathcal{O}}, \qquad \psi \mapsto s_\mu(\psi) = \widetilde{\psi},$$

called the *sewing map*. We denote $\Omega^0 = \Omega \setminus \{0\}$.

We recall that the sheaf $\mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})$ over $\mathcal{B}$, as well as its pull-back to the product $\mathcal{B} \times \Omega^0$ under the first projection, is equipped with the WZW-connection (see section 3.2). On the other hand, the restriction of the sheaf $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})$ to $\mathcal{B} \times \Omega^0$, which is the open subset of $\mathcal{B} \times \Omega$ parameterizing smooth curves, is also equipped with the WZW-connection. The main result of this section (Theorem 6.5) says that the sewing map s_μ is projectively flat for both connections. We first need to recall the following

Theorem 6.2 ([TUY] Theorem 6.2.2)**.** *For any section $\psi \in \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})$ the multi-valued formal power series $\widehat{\psi} = \tau^{\Delta_\mu}\widetilde{\psi}$ has the following properties :*

(1) *it satisfies the relation*

$$\nabla_{\tau\frac{d}{d\tau}}(\widehat{\psi}) = 0 \qquad \text{mod } \mathcal{O}_{\mathcal{B}\times\Omega}\widehat{\psi}.$$

(2) *for any $b \in \mathcal{B}$, the power series $\widetilde{\psi}_b$ converges.*

(3) *if $\mathcal{B}$ is compact, there exists a non-zero positive real number r such that the power series $\widetilde{\psi}$ defines a holomorphic section of $\mathcal{V}^{\dagger}_{l,\vec{\lambda}}(\mathcal{F})$ over $\mathcal{B} \times D_r$, where $D_r \subset \Omega$ is the open disc centered at 0 with radius r.*

Proof. Only part (3) is not proved in [TUY] Theorem 6.2.2. Consider a point $a \in \mathcal{B}$. We choose holomorphic coordinates $u_1, \ldots, u_m$ centered at the point $a \in \mathcal{B}$. Locally around the point $a \in \mathcal{B}$ the section $\widetilde{\psi}$ can be expanded as a $\mathcal{H}^{\dagger}_{\vec{\lambda}}$-valued power series in the $m+1$ variables $u_1, u_2, \ldots, u_m, \tau$. Given a second point $b \neq a$ with coordinates $b = (b_1, \ldots, b_m)$ with $b_i \neq 0$, we know by part (2) that $\widetilde{\psi}_b$ converges if $|\tau| \leq \rho$ for some real ρ. By the general theory of functions in several complex variables (see e.g. [O] Proposition 1.2) we deduce that $\widetilde{\psi}_c$ converges for $|\tau| < \rho$ and for any $c = (c_1, \ldots, c_m)$ such that $|c_i| < |b_i|$. Therefore, there exists for any $a \in \mathcal{B}$ a polydisc Δ_a around a and a real number r_a such that the radius of convergence of the series $\widetilde{\psi}_c$ for any $c \in \Delta_a$ is at least r_a. By considering the covering of $\mathcal{B}$ by the polydiscs Δ_a and by the fact that $\mathcal{B}$ is compact, we then obtain the desired non-zero real number r. $\square$

Remark 6.3. We note that the statement given in [TUY] Theorem 6.2.2 says that there exists a vector field $\vec{\ell}$ over the family of curves $\mathcal{C}$ such that

$$\left(-\tau \frac{d}{d\tau} + T[\vec{\ell}]\right).\widehat{\psi} = 0 \qquad \mathrm{mod}\ \mathcal{O}_{\mathcal{B}\times\Omega}\widehat{\psi},$$

which is equivalent to the above statement using the property $\theta(\vec{\ell}) = -\tau\frac{d}{d\tau}$. This last equality is actually proved in [TUY] Corollary 6.1.4, but there is a sign error. The correct formula of [TUY] Corollary 6.1.4 is $\theta(\vec{\ell}) = -\tau\frac{d}{d\tau}$, which is obtained by writing the 1-cocycle $\theta_{12}(u,\tau) = \tilde{\ell}'_{u,\tau|U_2} - \tilde{\ell}_{u,\tau|U_1}$.

Remark 6.4. By making the base change $\nu^j = \tau$, where j is the denominator of the trace anomaly Δ_μ, we obtain a section $\widehat{\psi} \in \mathcal{V}^{\dagger}_{l,\vec{\lambda}}(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{B}\times\Omega}} \mathcal{O}_{\mathcal{B}}[[\nu]]$ satisfying $\nabla_{\nu\frac{d}{d\nu}}(\widehat{\psi}) = 0 \ \mathrm{mod}\ \mathcal{O}_{\mathcal{B}\times\Omega}\widehat{\psi}$.

The next result says that the sewing map is projectively flat.

Theorem 6.5. *For any $\mu \in P_l$ and any $\vec{\lambda} \in (P_l)^n$ the restriction of the sewing map s_μ to the open subset $\mathcal{B} \times \Omega^0$*

$$s_\mu : \mathcal{V}^{\dagger}_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}}) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}\times\Omega^0} \longrightarrow \mathcal{V}^{\dagger}_{l,\vec{\lambda}}(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{B}\times\Omega^0}} \widehat{\mathcal{O}}$$

is projectively flat for the WZW connections on both sheaves of conformal blocks.

Proof. We need to check that $\nabla_D(\widetilde{\psi}) = 0 \ \mathrm{mod}\ \mathcal{O}_{\mathcal{B}\times\Omega^0}\widetilde{\psi}$ if $\nabla_D(\psi) = 0 \ \mathrm{mod}\ \mathcal{O}_{\mathcal{B}\times\Omega^0}\psi$ for any vector field D over $\mathcal{B} \times \Omega^0$. By $\mathcal{O}_{\mathcal{B}\times\Omega^0}$-linearity of the connection, it suffices to check the following two points:

(1) $\nabla_{\frac{d}{d\tau}}(\widetilde{\psi}) = 0 \ \mathrm{mod}\ \mathcal{O}_{\mathcal{B}\times\Omega^0}\widetilde{\psi}$ for any section ψ.

(2) $\nabla_\partial(\widetilde{\psi}) = 0 \ \mathrm{mod}\ \mathcal{O}_{\mathcal{B}\times\Omega^0}\widetilde{\psi}$ if $\nabla_\partial(\psi) = 0 \ \mathrm{mod}\ \mathcal{O}_{\mathcal{B}}\psi$ for any vector field ∂ on $\mathcal{B}$.

Part (1) is an immediate corollary of the previous Theorem 6.2 (1). Note that $\nabla_{\frac{d}{d\tau}}(\psi) = 0$ for any section ψ over $\mathcal{B}$.

We now prove part (2). We start with a lemma, which is an analogue of [U] Lemma 5.3.1.

Lemma 6.6. *Let $b \in \mathcal{B}$ and let ∂ be a vector field in some neighbourhood U of b. If we choose $U \subset \mathcal{B}$ sufficiently small, then there exist local coordinates $(u_1, \ldots, u_m, z)$ (resp. $(u_1, \ldots, u_m, w)$)*

of a neighbourhood X (resp. Y) of $s_{n+1}(U) \subset \widetilde{\mathcal{C}}_{|U}$ (resp. $s_{n+2}(U) \subset \widetilde{\mathcal{C}}_{|U}$) and a vector field $\vec{\ell}$ over $\widetilde{\mathcal{C}}_{|U}$, which is constant along the fibers

$$\vec{\ell} \in H^0(\widetilde{\mathcal{C}}_{|U}, \Theta'_{\widetilde{\mathcal{C}}}(* \sum_{i=1}^{n} s_i(\mathcal{B}))_{\widetilde{\pi}})$$

and which satisfy the following conditions :

(1) *the sections s_{n+1} and s_{n+2} are given by the mappings*

$$s_{n+1} : (u_1, \ldots, u_m) \mapsto (u_1, \ldots, u_m, 0) = (u_1, \ldots, u_m, z)$$

$$s_{n+2} : (u_1, \ldots, u_m) \mapsto (u_1, \ldots, u_m, 0) = (u_1, \ldots, u_m, w)$$

(2) $\vec{\ell}_{|X} = z\frac{d}{dz} + \partial$, $\vec{\ell}_{|Y} = -w\frac{d}{dw} + \partial$. *In particular, $\theta(\vec{\ell}) = \partial$, i.e. $\vec{\ell}$ projects onto the vector field ∂. Here θ denotes the projection on the horizontal component, see* (4).

Proof. The proof follows the lines of the proof of [U] Lemma 5.3.1.

For a small neighbourhood U of b, we choose $(u_1, \ldots, u_m, x)$ and $(u_1, \ldots, u_m, y)$ local coordinates in $\widetilde{\pi}^{-1}(U)$ satisfying condition (1) of the Lemma. We denote by $\widetilde{\pi}' : \widetilde{\mathcal{C}}' \longrightarrow \mathcal{B}$ the family of nodal curves parameterized by $\mathcal{B}$ obtained from the family $\widetilde{\mathcal{C}}$ by identifying the two divisors $s_{n+1}(\mathcal{B})$ and $s_{n+2}(\mathcal{B})$. Note that we have a sequence of maps over $\mathcal{B}$

$$\widetilde{\mathcal{C}} \xrightarrow{\nu} \widetilde{\mathcal{C}}' \longrightarrow \mathcal{C}_{|\mathcal{B}\times\{0\}}.$$

By choosing U small enough, we can lift the vector field ∂ over $U \subset \mathcal{B}$ to a vector field $\vec{\ell}$ over $\widetilde{\mathcal{C}}'_{|U}$ which is constant along the fibers of $\widetilde{\mathcal{C}}'_{|U} \to U$ and has poles only at $\mathcal{S}_{|U}$, i.e. lies in $\Theta'_{\widetilde{\mathcal{C}}'}(*\mathcal{S})_{\widetilde{\pi}'}$. The inclusion

$$\Theta'_{\widetilde{\mathcal{C}}'}(*\mathcal{S})_{\widetilde{\pi}'} \hookrightarrow \nu_* \Theta'_{\widetilde{\mathcal{C}}}(*\mathcal{S} - s_{n+1}(\mathcal{B}) - s_{n+2}(\mathcal{B}))_{\widetilde{\pi}}$$

allows us to see $\vec{\ell}$ as a vector field over $\widetilde{\mathcal{C}}_{|U}$ having the property

$$\frac{da}{dx}(u, 0) + \frac{db}{dy}(u, 0) = 0,$$

where the functions $a(u, x)$ and $b(u, y)$ are defined by the expressions of the restriction of $\vec{\ell}$ to the neighboorhoods X and Y

$$\vec{\ell}_{|X} = a(u, x)\frac{d}{dx} \text{ and } \vec{\ell}_{|Y} = b(u, y)\frac{d}{dy}.$$

Note that $a(u, 0) = 0$ and $b(u, 0) = 0$ for any $u \in U$. The rest of the proof then goes as in [U] Lemma 5.3.1 or [TUY] Lemma 6.1.2. □

Let $b \in \mathcal{B}$ and let ∂ be a vector field in some neighbourhood U of b. Taking U sufficiently small, we can lift the projective connection ∇ on the sheaf $\mathcal{V}^{\dagger}_{l,\vec{\lambda},\mu,\mu^{\dagger}}(\widetilde{\mathcal{F}})$ over U to a connection. We consider the vector field $\vec{\ell}$ constructed in Lemma 6.6. Then for a local section ψ over U the equation

$$\nabla_{\partial}(\psi) = 0 \bmod \mathcal{O}_U \psi$$

is equivalent to the equation

$$\partial\psi + T[\vec{\ell}]\psi + a\psi = 0 \qquad (8)$$

for some local section a of $\mathcal{O}_U$. We will take as local coordinates $\xi_{n+1} = z$ and $\xi_{n+2} = w$ around the divisors $s_{n+1}(U)$ and $s_{n+2}(U)$, as introduced in Lemma 6.6. Then the image of $\vec{\ell}$ under p can be written

$$\vec{\ell} = (\ell_1 \frac{d}{d\xi_1}, \cdots, \ell_n \frac{d}{d\xi_n}, \xi_{n+1} \frac{d}{d\xi_{n+1}}, -\xi_{n+2} \frac{d}{d\xi_{n+2}})$$

Since $T[\xi_i \frac{d}{d\xi_i}] = L_0$ acting on the i-th component of the tensor product, we obtain the following decomposition

$$T[\vec{\ell}] = \sum_{i=1}^{n} T[\ell_i] + L_0^{(n+1)} - L_0^{(n+2)},$$

where the exponent (i) of the Virasoro operator L_0 denotes an action on the i-th component.

For any non-negative integer d we then project equation (8) via the map π_d defined in (7) into $\mathcal{H}^\dagger_{\vec{\lambda}}$, which leads to

$$\partial \psi_d + \sum_{i=1}^{n} T[\ell_i]\psi_d + \pi_d(L_0^{(n+1)}\psi) - \pi_d(L_0^{(n+2)}\psi) + a\psi_d = 0.$$

We have the equalities $\pi_d(L_0^{(n+1)}\psi) = (\Delta_\mu + d)\psi_d$ and $\pi_d(L_0^{(n+2)}\psi) = (\Delta_{\mu^\dagger} + d)\psi_d$. Hence both terms cancel, since $\Delta_\mu = \Delta_{\mu^\dagger}$. This leads to the equations for any d

$$\partial \psi_d + \sum_{i=1}^{n} T[\ell_i]\psi_d + a\psi_d = 0. \tag{9}$$

Multiplying (9) with τ^d and summing over d, we obtain the equation

$$\partial \widetilde{\psi} + \sum_{i=1}^{n} T[\ell_i]\widetilde{\psi} + a\widetilde{\psi} = 0. \tag{10}$$

Note that $\partial(\psi_d \tau^d) = (\partial \psi_d)\tau^d$, since the vector field ∂ comes from $\mathcal{B}$.

The vector field $\vec{\ell}$ over $\widetilde{\mathcal{C}}_{|U}$ determines a vector field $\vec{m}$ over the family of smooth curves $\mathcal{C}_{|U\times\Omega^0}$ as follows. We fix a point $b \in \mathcal{B}$ and a non-zero complex number τ with $|\tau| < 1$. The smooth curve $\mathcal{C}_{(b,\tau)}$ is obtained from the curve $\widetilde{\mathcal{C}}_b$ by removing the two closed discs D_{n+1} and D_{n+2} centered at $s_{n+1}(b)$ and $s_{n+2}(b)$ with radius $|\tau|$, and by identifying in the open curve $\widetilde{\mathcal{C}}_b \setminus (D_{n+1} \cup D_{n+2})$ the two annuli

$$A_{n+1} = \{p \in \widetilde{\mathcal{C}}_b \ : \ |\tau| < |z(p)| < 1\} \text{ and } A_{n+2} = \{p \in \widetilde{\mathcal{C}}_b \ : \ |\tau| < |w(p)| < 1\}$$

according to the relation

$$zw = \tau.$$

Under this identification, we see that the two restrictions of vector fields $\vec{\ell}_{|\{b\}\times A_{n+1}}$ and $\vec{\ell}_{|\{b\}\times A_{n+2}}$ correspond (since $z\frac{d}{dz} = -w\frac{d}{dw}$) and thus define a vector field $\vec{m}$ over $\mathcal{C}_{(b,\tau)}$, which has poles only at the n points $s_1(b), \ldots, s_n(b)$. Moreover the Laurent expansion of $\vec{m}$ at $s_1(b), \ldots, s_n(b)$ coincide with the Laurent expansion of $\vec{\ell}$. For the construction in a family, see [U] section 5.3. Hence $\theta(\vec{m}) = \partial$ and $p(\vec{m}) = (\ell_1 \frac{d}{d\xi_1}, \cdots, \ell_n \frac{d}{d\xi_n})$. So equation (10) can be written as

$$\nabla_\partial \widetilde{\psi} = \partial \widetilde{\psi} + T[\vec{m}]\widetilde{\psi} = 0 \text{ mod } \mathcal{O}_{\mathcal{B}\times\Omega^0}\widetilde{\psi}.$$

The last equation means that $\widetilde{\psi}$ is a projectively flat section for the WZW connection. $\square$

From now on we assume that $\mathcal{B}$ is compact. Since by Theorem 6.2 (3) the formal power series $\widetilde{\psi}$ determines a holomorphic section over $\mathcal{B} \times D_r$ we can choose a complex number $\tau_0 \neq 0$ with $|\tau_0| < r$ and evaluate $\widetilde{\psi}$ at τ_0. This gives a section $\widetilde{\psi}(\tau_0)$ of the conformal block $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0}) = \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F})_{|\mathcal{B}\times\{\tau_0\}}$.

Moreover, using the factorization rules (see e.g. [TUY] Theorem 6.2.6 or [U] Theorem 4.4.9) we obtain by summing over all dominant weights $\mu \in P_l$ an $\mathcal{O}_{\mathcal{B}}$-linear isomorphism

$$\oplus s_\mu(\tau_0) : \bigoplus_{\mu \in P_l} \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}}) \xrightarrow{\sim} \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0}),$$

which is projectively flat for the WZW connections on both sheaves over $\mathcal{B}$ by Theorem 6.5. We fix a base point $b \in \mathcal{B}$, which gives a direct sum decomposition

$$\bigoplus_{\mu \in P_l} \mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})_b \xrightarrow{\sim} \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})_b. \tag{11}$$

We denote by D the subgroup of $\mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})_b)$ consisting of projective linear maps preserving the direct sum decomposition (11) and by $p_\mu : \mathrm{D} \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})_b)$ the projection onto the summand corresponding to $\mu \in P_l$.

The next proposition is an immediate consequence of the fact that the maps $s_\mu(\tau_0)$ are projectively flat.

Proposition 6.7. *With the above notation we have for any $\mu \in P_l$ and any $\vec{\lambda} \in (P_l)^n$*

(1) *the monodromy representation of the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})$ over $\mathcal{B} \times \{\tau_0\}$ takes values in the subgroup* D*, i.e.,*

$$\rho_{l,\vec{\lambda}} : \pi_1(\mathcal{B}, b) \longrightarrow \mathrm{D} \subset \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})_b).$$

(2) *we have a commutative diagram*

$$\begin{array}{ccc} \pi_1(\mathcal{B}, b) & \xrightarrow{\rho_{l,\vec{\lambda}}} & \mathrm{D} \\ \big\downarrow = & & \big\downarrow p_\mu \\ \pi_1(\mathcal{B}, b) & \xrightarrow{\rho_{l,\vec{\lambda},\mu,\mu^\dagger}} & \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathcal{F}})_b) \end{array}$$

In the proof of the main theorem we will use the above proposition for a slightly more general family $\mathcal{F}$ of n-pointed connected projective curves. We shall assume that $\mathcal{F}$ satisfies the two conditions:

(1) the curve $\mathcal{C}_{(b,\tau)}$ is smooth if $\tau \neq 0$.
(2) the curve $\mathcal{C}_{(b,0)}$ has exactly m nodes.

The desingularizing family $\widetilde{\mathcal{F}}$ will thus be a $n + 2m$-pointed family.

Remark 6.8. An example of a family $\mathcal{F}$ satisfying the above conditions is given in section 6.1.2.

Given an m-tuple $\vec{\mu} = (\mu_1, \ldots, \mu_m) \in (P_l)^m$ of dominant weights, we denote $\vec{\mu}^\dagger = (\mu_1^\dagger, \ldots, \mu_m^\dagger) \in (P_l)^m$. The next proposition is shown along the same lines as Proposition 6.7. First we note that there is a decomposition

$$\bigoplus_{\mu \in (P_l)^m} \mathcal{V}^\dagger_{l,\vec{\lambda},\vec{\mu},\vec{\mu}^\dagger}(\widetilde{\mathcal{F}})_b \xrightarrow{\sim} \mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})_b. \tag{12}$$

We denote by D the subgroup of $\mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})_b)$ preserving the above decomposition.

Proposition 6.9. *With the above notation we have for any $\vec{\mu} \in (P_l)^m$ and any $\vec{\lambda} \in (P_l)^n$*

(1) *the monodromy representation of the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})$ over $\mathcal{B} \times \{\tau_0\}$ takes values in the subgroup* D, *i.e.,*
$$\rho_{l,\vec{\lambda}} : \pi_1(\mathcal{B}, b) \longrightarrow \mathrm{D} \subset \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda}}(\mathcal{F}_{\tau_0})_b).$$

(2) *we have a commutative diagram*
$$\begin{array}{ccc} \pi_1(\mathcal{B}, b) & \xrightarrow{\rho_{l,\vec{\lambda}}} & \mathrm{D} \\ \Big\downarrow = & & \Big\downarrow p_\mu \\ \pi_1(\mathcal{B}, b) & \xrightarrow{\rho_{l,\vec{\lambda},\vec{\mu},\vec{\mu}^\dagger}} & \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,\vec{\lambda},\vec{\mu},\vec{\mu}^\dagger}(\widetilde{\mathcal{F}})_b) \end{array}$$

6.3. **Proof of the Theorem.** We will now prove the theorem stated in the introduction. We know by [La] assuming[1] $g \geq 2$ that there is a projectively flat isomorphism between the two projectivized vector bundles
$$\mathbf{P}\mathcal{Z}_l \xrightarrow{\sim} \mathbf{P}\mathcal{V}^\dagger_{l,\emptyset}$$
equipped with the Hitchin connection and the WZW connection respectively. Here $\mathcal{V}^\dagger_{l,\emptyset}$ stands for the sheaf of conformal blocks $\mathcal{V}^\dagger_{l,0}(\mathcal{F})$ associated to the family $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1)$ of curves with one point labeled with the trivial representation $\lambda_1 = 0$ (propagation of vacua).

We consider the family of hyperelliptic curves $\mathcal{F} = \mathcal{F}_g^{hyp}$ defined in section 6.1.2 for some α with $|\alpha| > 1$. The family $\widetilde{\mathcal{F}}$ which desingularizes the nodal curves $\mathcal{F}_{|\mathcal{B}\times\{0\}}$ is a family of $(2g+1)$-pointed rational curves with points
$$\infty, 1, -1, u, -u, 3\alpha, -3\alpha, \ldots, g\alpha, -g\alpha.$$
We then deduce from Proposition 6.9 (2) applied to the family $\mathcal{F} = \mathcal{F}_g^{hyp}$ with $n = 1$, $m = g$ and the choice of weights $\lambda_1 = 0$ and $\mu_1 = \cdots = \mu_g = \varpi$, where we associate the weight 0 to ∞ and the weight ϖ to the remaining $2g$ points (note that $\varpi = \varpi^\dagger$), that it suffices to show that the monodromy representation
$$\pi_1(\mathcal{B}, b) \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,0,\varpi,\ldots,\varpi}(\widetilde{\mathcal{F}})_b)$$
has an element of infinite order in its image.

In order to show the last statement we consider the family of rational curves $\mathcal{F}_{2g+1}^{rat}$ defined in section 6.1.1. The family $\widetilde{\mathcal{F}}$ which desingularizes the nodal curves $\mathcal{F}_{|\mathcal{B}\times\{0\}}$ is a family of

[1]In fact in [La] one makes the assumption $g \geq 3$ for simplicity. It can be shown that the isomorphism also holds for $g = 2$.

$(2g+3)$-pointed rational curves consisting of the disjoint union of two projectve lines with 5 points $0, 1, -1, u, -u$ on one projective line and $2g-2$ points $0, \infty, 3, -3, \ldots, g, -g$ on the second projective line.

Next we observe that the conformal block for the projective line with $2g-2$ marked points $0, \infty, 3, -3, \ldots, g, -g$ with the zero weight at the points 0 and ∞ and the weight ϖ at the other $2g-4$ points is non-zero. This follows from an iterated use of the propagation of vacua, the factorization rules and from the fact that $\dim \mathcal{V}_{l,\varpi,\varpi}(\mathbf{P}^1) = 1$.

The previous observation together with Corollary 5.5 then implies that the family $\widetilde{\mathcal{F}}$ of $(2g+2)$-pointed curves with weights $0, \varpi, \varpi, \varpi, \varpi$ on the 5 points $0, 1, -1, u, -u$ of the first projective line and weights $0, 0, \varpi, \ldots, \varpi$ on the $2g-2$ points $0, \infty, 3, -3, \ldots, g, -g$ on the $2g-2$ points on the second projective line has infinite monodromy. We then deduce from Proposition 6.7 (2) applied to the family $\mathcal{F} = \mathcal{F}^{rat}_{2g+1}$ with $\vec{\lambda} = (\varpi, \ldots, \varpi)$ and $\mu = 0$ that the monodromy representation

$$\pi_1(\mathcal{B}, b) \longrightarrow \mathbf{P}\mathrm{GL}(\mathcal{V}^\dagger_{l,0,\varpi,\ldots,\varpi}(\widetilde{\mathcal{F}})_b)$$

has an element of infinite order in its image, which completes the proof.

Remark 6.10. For the convenience of the reader we recall that we have taken the family of smooth hyperelliptic curves given by the affine equation (6) for two complex numbers α and τ with α^{-1} and τ sufficiently small — note that α^{-1} and τ measure the size of the domain where the sewing elements $\hat{\psi}$ for the two families $\mathcal{F}^{rat}_{2g+1}$ and $\mathcal{F}^{hyp}_g$ converge. The parameter u varies in $\mathcal{B}$ as defined in (5). Then the loop $\xi \in \pi_1(\mathcal{B}, i)$ which starts at i and goes once around the points -1 and 0 has monodromy of infinite order.

Remark 6.11. The previous argument, which proves the theorem for the Lie algebra $\mathfrak{sl}(2)$, fails when considering the Lie algebras $\mathfrak{sl}(r)$ with $r > 2$. The main reason is the fact $\varpi^\dagger \neq \varpi$ for $r > 2$, where ϖ is the first fundamental weight.

7. Finiteness of the monodromy representation in genus one

In this section we collect for the reader's convenience some existing results on the monodromy representation on the conformal blocks associated to the Lie algebra $\mathfrak{sl}(2)$ for a family of one-marked elliptic curves labeled with the trivial representation. We consider the upper half plane $\mathbf{H} = \{\omega \in \mathbf{C} \mid \mathrm{Im}\, \omega > 0\}$ with the standard action of the modular group $\mathbf{P}\mathrm{SL}(2, \mathbf{Z})$, which is generated by the two elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{satisfying } S^2 = (ST)^3 = e.$$

Let $\mathcal{F}$ denote the universal family of elliptic curves parameterized by $\mathbf{H}$. We denote by $\mathcal{V}^\dagger_{l,0}(\mathcal{F})$ the sheaf of conformal blocks of level l with trivial representation at the origin. The sheaf $\mathcal{V}^\dagger_{l,0}(\mathcal{F})$ has rank $l+1$ and for each $\lambda \in P_l$ we obtain by the sewing procedure a section $\widehat{\psi}_\lambda$ over $\mathbf{H}$ given by the formal series

$$\langle \widehat{\psi}_\lambda(\omega) | \phi \rangle = \tau^{\Delta_\lambda} \sum_{d=0}^{\infty} \langle \psi | \gamma_d \otimes \phi \rangle \tau^d, \qquad \text{with } \ \tau = \exp(2i\pi\omega),$$

where ψ is the unique (up to a multiplicative scalar) section of $\mathcal{V}^\dagger_{l,\lambda,\lambda,0}(\mathbf{P}^1)$ and ϕ is any element in $\mathcal{H}_\lambda$. Because of Theorem 6.2 the $l+1$ sections $\widehat{\psi}_\lambda$ are projectively flat for the projective

WZW connection on $\mathcal{V}^{\dagger}_{l,0}(\mathcal{F})$ and are linearly independent by the factorization rules (11). Note that this decomposition of the sheaf $\mathcal{V}^{\dagger}_{l,0}(\mathcal{F})$ into a sum of rank-1 subsheaves corresponds to a degeneration to the nodal elliptic curve given by $\mathrm{Im}\,\omega \to \infty$, or equivalently $\tau = \exp(2i\pi\omega) \to 0$. Moreover by evaluating the sections $\widehat{\psi}_\lambda$ at the highest weight vector $\phi = v_\lambda \in \mathcal{H}_\lambda$ we obtain analytic functions $\chi_\lambda(\omega) = \langle \widehat{\psi}_\lambda(\omega) | v_\lambda \rangle$, which correspond to the character of the representations $\mathcal{H}_\lambda$:

$$\chi_\lambda(\omega) = \sum_{d=0}^{\infty} \sum_{i=1}^{m_d} (v_i(d)|v^i(d)) \tau^{\Delta_\lambda + d} = \sum_{d=0}^{\infty} \dim \mathcal{H}_\lambda(d) \tau^{\Delta_\lambda + d} = \mathrm{tr}_{\mathcal{H}_\lambda}(\tau^{L_0}),$$

see e.g. [U] equation (4.3.1). This shows that in the genus one case the local system given by the conformal blocks with trivial marking equipped with the WZW connection coincides with the local system given by the characters $\chi_\lambda(\omega)$. Moreover the monodromy action of the modular group $\mathbf{P}\mathrm{SL}(2,\mathbf{Z})$ on the vector space spanned by the characters $\{\chi_\lambda(\omega)\}_{\lambda \in P_l}$ has been determined.

Proposition 7.1 ([GW])**.** *The monodromy representation*

$$\rho_l : \mathbf{P}\mathrm{SL}(2,\mathbf{Z}) \longrightarrow \mathbf{P}\mathrm{GL}(l+1)$$

is given by the two unitary matrices $\rho(S)$ and $\rho(T)$

$$\begin{aligned} \rho(S)_{jk} &= \sqrt{\frac{2}{l+2}} \sin\left(\frac{\pi jk}{l+2}\right), \\ \rho(T)_{jk} &= \delta_{jk} \exp\left(i\pi\left(\frac{j^2}{2(l+2)} - \frac{1}{4}\right)\right). \end{aligned}$$

With this notation the main statement of this section is the following

Theorem 7.2. *The image of the representation ρ_l is finite.*

Proof. Using the explicit expression of the matrix $\rho_l(U)$ for any element $U \in \mathbf{P}\mathrm{SL}(2,\mathbf{Z})$ computed in [J] section 2, it is shown in [G] section 2 that the matrix $\rho_l(U)$ has all its entries in the set $\frac{1}{2(l+2)}\mathbf{Z}[\exp(\frac{i\pi}{4(l+2)})]$. Since moreover the representation ρ_l is unitary, we may deduce finiteness along the same lines as in [G] proof of corollary. □

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